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STATISTICS OF SATELLITES THAT ARE LAUNCHED AT REGULAR INTERVALS AND HAVE AN EXPONENTIAL DISTRIBUTION OF LIFETIMES

by

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1. Introduction

This note describes a method for computing the answers to the following questions concerning the satellites described in the title:

(a) What is the probability of having m live satellites in orbit immediately after the nth launch?

(b) What is the probability of having m live satellites in orbit for the first time immediately after the nth launch?

Question (b) arises, for example, when one is planning the task of putting up an initial set of m communication satellites.

2. The Probability of Having m Live Satellites at the nth Launch*

Let the satellites be launched one at a time, let Δt be the time between consecutive launches, let p be the probability that a launch will be successful, and let the lifetimes of successfully launched satellites be distributed exponentially and have the mean value $\bar{t}$. The probability that a successfully launched satellite will survive until the scheduled launch of the next satellite is then

$$c = e^{-\Delta t / \bar{t}}. \tag{1}$$

Furthermore, if there are exactly j live satellites at one of the launches, then the probability that exactly i of them will survive until the next launch is

$$\frac{j!}{i!\,(j-i)!} c^i (1-c)^{j-i}. \tag{2}$$

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* For brevity, we write "live satellites" for "live satellites in the proper orbits" and "at the nth launch" for "immediately after the nth launch."

Let A_{ij} be the probability that we will have exactly i live satellites at a launch if we had exactly j live satellites at the preceding launch. When we write

$$q = 1-p, \quad d = 1-c, \tag{3}$$

the formula for A_{ij} is

$$A_{ij} = \frac{j!}{i!\,(j-i)!} c^i d^{j-i} q \quad + \quad \frac{j!}{(i-1)!\,(j-i+1)!} c^{i-1} d^{j-i+1} p\,. \tag{4}$$

The first term on the right-hand side of (4) is the probability that i of the already available j satellites survive and the next launch fails; the second term is the probability that only $i-1$ of the already available j satellites survive but the next launch succeeds. Note the special cases

$$A_{ij} = d^j q\,, \qquad \text{if } i = 0, \tag{5}$$

$$A_{ij} = c^j p\,, \qquad \text{if } i = j+1\,, \tag{6}$$

$$A_{ij} = 0\,, \qquad \text{if } i > j+1. \tag{7}$$

The "transition probabilities" A_{ij} form the infinite matrix

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots \\ 0 & A_{21} & A_{22} & A_{23} & \cdots \\ 0 & 0 & A_{32} & A_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{8}$$

More explicitly,

$$A = \begin{pmatrix} q & dq & d^2q & d^3q & \cdots \\ p & cq+dp & 2cdq+d^2p & 3cd^2q+d^3p & \cdots \\ 0 & cp & c^2q+2cdp & 3c^2dq+3cd^2p & \cdots \\ 0 & 0 & c^2p & c^3q+3c^2dp & \cdots \\ 0 & 0 & 0 & c^3p & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} . \qquad (9)$$

Let u_m^k be the probability of having exactly m live satellites at the kth launch. The probability distribution of live satellites at this launch can then be described by the vector

$$u^k = \begin{pmatrix} u_0^k \\ u_1^k \\ u_2^k \\ \cdots \\ \cdots \end{pmatrix} . \qquad (10)$$

The top component of u^k is the probability of having no live satellites at the kth launch, its second component is the probability of having exactly one live satellite at that launch, and so on. Note that

$$u_m^k = 0 \text{ if } m>k. \qquad (11)$$

The vector describing the probability distribution of live satellites at the (k+1)th launch is

$$u^{k+1} = \begin{pmatrix} u_0^{k+1} \\ u_1^{k+1} \\ u_2^{k+1} \\ \cdots \\ \cdots \end{pmatrix} . \tag{12}$$

Our immediate problem is to show how the vector (12) can be computed from the vector (10).

Consider the probability, u_i^{k+1}, of having exactly i live satellites at the (k+1)th launch. If we knew that we certainly had exactly j live satellites at the kth launch, the probability of having exactly i live satellites at the (k+1)th launch would be A_{ij}. However, all we know is that the probability of having had exactly j live satellites at the kth launch is u_j^k. Consequently the possibility that we had exactly j live satellites at the kth launch contributes to u_i^{k+1} the term $A_{ij}u_j^k$; similarly, the possibility that we had j' live satellites at the kth launch contributes to u^{k+1} the term $u_{j'}^k A_{ij'}$, and so on.

Accordingly,

$$u_i^{k+1} = \sum_{j=0}^{k} A_{ij} u_j^k . \tag{13}$$

In view of (11), Equation (13) implies that the vector u^{k+1} is the product of the matrix A and the vector u^k; that is,

$$u^{k+1} = Au^k , \tag{14}$$

although only a finite portion of A is in fact needed on the right-hand side of (14). It follows that $u^{k+2} = A^2 u^k$, $u^{k+3} = A^3 u^k$, and so on. As we shall now illustrate, Equation (14) provides a method for answering the question (a) stated in the Introduction.

An Example. Let us use (14) to compute the probabilities pertaining to the situation at the second launch. For this purpose it is convenient to speak of a zeroth launch, which is actually no launch at all, so that the probability of having no live satellites just after it is unity and the vector u^0 has the form shown in (15). Multiplying this u^0 by the matrix A, given by (9), we get the vector u^1 in (15); and multiplying this u^1 by A, we get the vector u^2.

$$u^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \cdots \end{pmatrix} \qquad u^1 = Au^0 = \begin{pmatrix} q \\ p \\ 0 \\ 0 \\ \cdots \end{pmatrix}, \quad u^2 = Au^1 = \begin{pmatrix} q^2+dqp \\ pq+cqp+dp^2 \\ cp^2 \\ 0 \\ \cdots \end{pmatrix}. \qquad (15)$$

It follows that the probability of having no live satellites at the second launch is q^2+dqp, the probability of having one live satellite at that time is $pq+cqp+dp^2$, and the probability of having two live satellites at that time is cp^2. Note that to obtain these results we needed only a finite portion of A. In problems of this kind, the formal equation that gives the probabilities pertaining to the situation immediately after the nth launch is

$$u^n = A^n u^0 , \qquad (16)$$

although actually only a finite portion of A is involved.

Second Example. In the preceding example the vector u^2 was computed on the assumption that the outcome of the first launch was unknown. To illustrate a different situation, let us compute u^2 on the assumption that the first launch was successful, so that the probability of having one live satellite at the first launch was unity. The vector u^1 now has the form shown in (17), and multiplying it by A we get a new vector u^2. Comparing this vector with the u^2 in (15), we find that the probability of having no

live satellites at the second launch is now smaller, and the probability of

$$u^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \cdots \end{pmatrix}, \quad u^2 = Au^1 = \begin{pmatrix} dq \\ cq+dp \\ cp \\ 0 \\ \cdots \end{pmatrix} \qquad (17)$$

having two live satellites at this time is larger.

3. The "First Time" Problem

The planning of launches for a communication system that uses m satellites includes the consideration of two distinct tasks:

(1) to put up an initial set of m live satellites, and

(2) to maintain the system by replacing the satellites that fail after the initial set has been put up.

We shall deal here only with the first task and with the probability (denoted below by D_m^n) that this task will be completed in exactly n launches. In other words, we shall deal with the probability that exactly m live satellites will become available for the first time at exactly the nth launch.

To illustrate the distinction between the probabilities u_m^n and D_m^n, we assume that three live satellites are to be put up $(m = 3)$ and consider the case of six launches $(n = 6)$. One possible sequence of the numbers of live satellites available just after the first launch, just after the second launch, and so on, runs as follows:

$$1,\ 2,\ 0,\ 1,\ 2,\ 3 \text{ satellites} . \qquad (18)$$

Another possible sequence is

$$1,\ 2,\ 2,\ 3,\ 2,\ 3 \text{ satellites} . \qquad (19)$$

The probability u_3^6 takes into account both of these possibilities. But in the computation of D_3^6 the sequence (19) must be excluded, because it implies that the prescribed task would be completed in four (not six) launches.

In the study of the "first time" problem we shall denote the matrix of the transition probabilities by B rather than A, and the probability distribution vectors by v's rather than u's.

The necessary and sufficient conditions for obtaining m live satellites for the first time at the nth launch are:

(a) at none of the first n-1 launches must the number of live satellites exceed m-1;

(b) at the (n-1)th launch, m-1 live satellites must be available; and

(c) at the nth launch the number of live satellites must change from m-1 to m.

The First n-1 Launches. We shall now consider the first n-1 launches and shall take up the nth launch later as a separate problem.

The condition (a) implies that the v's whose superscripts are smaller than n can have at most m nonvanishing components. Therefore these v's (unlike the u's) can be treated as vectors having a finite number of components. For example, if n>3, the explicit form of v^3 is

$$v^3 = \begin{pmatrix} v_0^3 \\ v_1^3 \\ \cdots \\ v_{m-1}^3 \end{pmatrix} . \qquad (20)$$

The infinite matrix **A**, given by (9), includes transitions that violate the condition (a). To eliminate them, we replace **A** by the finite matrix

$$B = \begin{pmatrix} B_{00} & B_{01} & \cdots & B_{0,m-1} \\ B_{10} & B_{11} & \cdots & B_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{m-1,m-1} \end{pmatrix} \qquad (21)$$

where

$$B_{ij} = A_{ij}, \qquad i, j = 0, 1, \ldots, m-1 \; . \qquad (22)$$

Equation (14) must now be replaced by

$$v^{k+1} = Bv^k, \qquad k \leq n-2 \; . \qquad (23)$$

In particular, if all we know is the outcome of the zeroth launch, we have

$$v^{n-1} = B^{n-1}v^0 \; , \qquad (24)$$

where v^0 is the vector whose first component is unity and whose other components are zeros. But if one knows how many live satellites are available immediately after some actual launch, he would use the procedure illustrated in our second example concerning the u's.

Note that (23) takes care of all the transitions which are consistent with the condition (a) and ignores all the transitions which are not.

The nth Launch. We now turn to the conditions (b) and (c) on page 7. The probability that (b) will be satisfied is the element v^{n-1}_{m-1} of the vector v^{n-1}, computed from (23) or (24). In view of (6), the probability that (c) will be satisfied is $c^{m-1}p$. Therefore the formula for the probability (which we have denoted by D^n_m) that exactly m live satellites will become

available <u>for the first time</u> at exactly the nth launch is

$$D_m^n = v_{m-1}^{n-1} c^{m-1} p. \qquad (25)$$

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